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Stability Analysis of Switched Linear Systems with Locally Overlapped Switching Law

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Combining the common Lyapunov function method and the dwell time method, we propose stability analysis tools for switched linear systems evolving on the locally overlapped switching law. These systems exist naturally in many aerospace applications, such as separation dynamical systems involving rapid and discrete variations that are specified by the predetermined separation sequence. We show that switched linear systems with locally overlapped switching law are globally asymptotically stable provided that the dwell time on some subsystems is no smaller than a fixed positive constant. This result is also extended to the case of combination of the common Lyapunov function method and the average dwell time method. We validate the proposed stability analysis tools by an example and illustrate an application to full-envelope flight controller design.

I. Introduction

ANALYSIS and synthesis for switched systems have attracted increasing attention in the control community. As an important class of hybrid systems, switched systems are composed of a collection of subsystems described by differential/difference equations, together with a switching law that specifies the switching between the subsystems. Switched systems have a wide range of applications in physical and engineering systems, which are mainly due to numerous practical systems exhibiting a switched/hybrid nature and the growing use of computers in the control of physical plants [1,2]. In the aerospace field, many design problems can be viewed as the analysis and synthesis of the switched system, such as aircraft controller design [3], aircraft fault tolerant controller design [4], stability of spacecraft formation [5], angle of attack and normal acceleration limiter design [6], and modeling and control of flow systems [7].

Stability is a major and challenging issue in studying switched systems, because of complicated behavior caused by interaction between continuous and discrete dynamics [8]. For switched systems with the time-dependent switching law, the common Lyapunov function method and the average dwell time method are two primary stability analysis technologies. Under arbitrary switching laws, sharing a common Lyapunov function is a sufficient and necessary condition for the stability of a switched system in which the subsystems are all stable. However, it is deemed to be so conservative that only a few classes of switched systems with a special structure satisfy this condition, such as commuting systems [2]. An average dwell time method that is extended from the dwell time method is a quite useful stability analysis tool for switched systems [9]. The average dwell time method only restricts limited numbers of switching between a finite time interval and allows the possibility of switching fast when necessary and switching sufficiently slow later on for stability.

In this work, we combine the common Lyapunov function method and the (average) dwell time method to analyze the stability of switched linear systems with a locally overlapped switching law.

A switching law with the property of "locally overlapped" exists in many practical engineering problems. Take the workshop production process, for example, in which the order of the activated subsystems is predetermined [10], that is, we must activate subsystem 1, then activate subsystem 2, and then switch to subsystem 3 from subsystem 2, etc. In this case, the switching law 1-2-3 can be viewed as two locally overlapped parts, one is 1-2 and the other is 2-3, while the switched system is also partitioned into two locally overlapped groups corresponding to two locally overlapped switching sequences, and subsystem 2 can be viewed as the common subsystem of the two groups.

In aerospace applications, separation dynamical systems specified by a predetermined separation sequence can be viewed as switched systems and the switching law of such systems has the property of "locally overlapped." For example, in space-shuttle–rocket-booster separation, the solid rocket-booster power first terminates (event A), then thrust-vectoring control is commanded to null (event B), and then the booster separation motors are commanded to fire (event C) [11]. Because the switching law A-B-C can be regarded as two locally overlapped parts (A-B and B-C), the switched system corresponding to space-shuttle–rocket-booster separation dynamics is partitioned into two locally overlapped groups and the subsystem associated with event B is the common subsystem of the two groups. A similar situation exists in the successive landing phases of the aircraft (descent, flare, and then touchdown), during which the system switches dynamics several times [12,13]. Another typical example in the aerospace field is full-envelope flight controller design. We assume that the switched system constructed by several decoupled linear models, corresponding to finite operating points within the flight envelope, can describe full-envelope dynamics and that the switching law evolves upon aircraft altitude and Mach number. Considering that the altitude and Mach number of the aircraft could not vary discretely, the active subsystem corresponding to one operating point can only switch to the adjacent operating points, that is, the switching law is partitioned into several locally overlapped parts rather than arbitrarily assigned, which is similar to the workshop example mentioned earlier. Because of the locally overlapped property of the switching law within the flight envelope, the switched system is naturally partitioned into several locally overlapped groups.

Although the switching law with a locally overlapped property is a familiar and nontrivial phenomenon in engineering, there has been no effective stability analysis method for such switched systems yet. Especially in the aerospace field, to our knowledge, the stability of these systems has only been probed via extensive simulation. For such systems, a common Lyapunov function method is too conservative because the switching law is not arbitrary and it is unpractical that all subsystems share a common Lyapunov function with the increasing mode of a switched system. The (average) dwell

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time is a feasible tool to analyze switched systems with a locally overlapped switching law, but it is (average) dwell time restricted during a finite time interval. In our work we use a common Lyapunov function to guarantee the stability for each locally overlapped group of the switched system and use (average) dwell time to guarantee stability when switching occurs between different locally overlapped groups. We obtain effective stability analysis tools by virtue of such a combination. Sharing a common Lyapunov function, respectively, for each locally overlapped group assures stability under arbitrary switching, and it is much easier to be satisfied comparing to all subsystems sharing a common Lyapunov function. Also, the constraint on (average) dwell time for each subsystem is replaced by (average) duration in some common subsystems belonging to finite locally overlapped groups, which is obviously less conservative.

Note that a common Lyapunov function might not exist for some locally overlapped groups. In this case, we can divide the locally overlapped group into several sublocally overlapped groups and then check whether they share a common Lyapunov function or not. A special case of this management is that all groups are divided into individual subsystems and, thereby, the stability analysis method presented in this paper degenerates into the average dwell time method. Another manner to deal with this problem is repartitioning the switched system into another set of locally overlapped groups and then checking the existence of a common Lyapunov function.

The remainder of the paper is organized as follows. Section II illustrates some concepts relating to the switched systems evolving on a locally overlapped switching law and formulates the problem. Section III presents the stability analysis method with a combination of common Lyapunov function and dwell time. In Sec. IV, this method is extended to the combination of common Lyapunov function and average dwell time, and an application to full-envelope flight controller design is illustrated. Finally, conclusions are presented in Sec. V.

II. Problem Statement

A. Concepts Relating to Locally Overlapped Switching Law

A class of switched systems is given by

$$\dot{x}(t) = f_{\sigma(t)}(x(t), t) \tag{1}$$

where $x(t) \in \Re^n$ is the continuous state, the piecewise constant $\sigma(t)$ is the switching law and takes value discretely from a finite index set Ω at every switching, and the functions $f_{\sigma(t)}$ are assumed to be sufficiently regular (at least locally Lipschitz).

For a natural number k, let $\sim^k$ denote the set $\{1, 2, \cdots, k\}$.

Definition 1: The switching law $\sigma(t)$ is called *localizable* if there exist finite index sets Ω_i, $j \in \sim^k$ with the property that $\bigcup_{s=1}^{k} \Omega_s = \Omega$ and $\forall\ l \in \sim^k$, $\Omega_l \bigcap (\bigcup_{s=1, s\neq l}^{k} \Omega_s) \neq \emptyset$ such that $\forall\ t \geq 0$, it holds that $\sigma(t^+) \in \bigcup_{j\in\{w|\sigma(t)\in\Omega_w\}} \Omega_j$.

In Definition 1, we show that switched systems with a localizable switching law have two interesting properties. One property is that subsystems of the switched system can be partitioned into several locally overlapped groups. The other property is that the switching law evolves in a specific way of either switching within a locally overlapped group or switching to adjacent locally overlapped groups. The two properties permit us to study the switched system in an equivalent formulation, which is specified by the following Definition 2.

Definition 2: If the switching law $\sigma(t)$ evolving upon the switched system (1) is localizable, then we call $\sigma(t)$ a *locally overlapped switching law*. Further, the switched system (1) can be reformulated as Eq. (2), called a *localizable formulation*, in which $T^j \triangleq \{t|\sigma(t) \in \Omega_j\}$, $j \in \sim^k$ and $\sigma_j(t) \triangleq \sigma(t)$, $t \in T^j$.

$$\begin{cases} \dot{x}(t) = f_{\sigma_1(t)}(x(t), t), & \sigma_1(t) \to \Omega_1 \\ \dot{x}(t) = f_{\sigma_2(t)}(x(t), t), & \sigma_2(t) \to \Omega_2 \\ \qquad\vdots & \\ \dot{x}(t) = f_{\sigma_k(t)}(x(t), t), & \sigma_k(t) \to \Omega_k \end{cases} \tag{2}$$

The motivation for constructing a localizable formulation is to investigate the stability of the switched system by finite subswitched systems. The following definitions are used to detail the structure of the localizable formulation.

Definition 3: Given the localizable formulation (2) of the switched system (1), which evolves upon locally overlapped switching law $\sigma(t)$, we define

$$\dot{x}(t) = f_{\sigma_j(t)}(x(t), t), \qquad \sigma_j(t) \to \Omega_j, \qquad j \in \sim^k \tag{3}$$

as a *locally overlapped switched system* (LOSS) and therein define $\sigma_j(t)$ to be the *subswitching law*.

Definition 4: Given the localizable formulation (2) of the switched system (1), which evolves upon locally overlapped switching law $\sigma(t)$, we define

$$\dot{x}(t) = f_i(x(t), t), \qquad i \in \Omega_{\mathfrak{J}}^c = \bigcap_{\varepsilon\in\mathfrak{J}} \Omega_\varepsilon \tag{4}$$

to be the *subset $\mathfrak{J}$-based common subsystem*, where $\mathfrak{J}$ is any nonempty subset of $\sim^k$ with the property that $\Omega_{\mathfrak{J}}^c \neq \emptyset$.

Remark 1: The concept of LOSS is equivalent to a locally overlapped group, which is mentioned several times in the Introduction. The latter is convenient for expressing engineering problems and easily understood, whereas the former is exact for stability analysis of switched systems.

Remark 2: The mode in form (2) for the family (1) is not unique when the switching law is localizable. The number of possible modes increases as the complexity of the switching law or the number of subsystems increases.

B. Simple Example

For the second-order switched linear system with three subsystems given by

$$\dot{x}(t) = A_{\sigma(t)} x(t) \tag{5}$$

suppose that the switching law is localizable, which is depicted in Fig. 1. Therefore, the switched system (5) can be partitioned into two LOSSs and $\sigma_j(t)$, $j = 1, 2$ are two subswitching laws.

$$\text{Two LOSSs}: \begin{array}{ll} \dot{x}(t) = A_{\sigma_1(t)} x(t), & \sigma_1(t) \to \Omega_1 = \{1, 2\} \\ \dot{x}(t) = A_{\sigma_2(t)} x(t), & \sigma_2(t) \to \Omega_2 = \{2, 3\} \end{array} \tag{6}$$

and $\dot{x}(t) = A_2 x(t)$ is the subset $\{1, 2\}$-based common subsystem.

C. Problem Formulation

The objectives of our work are to address the following problems:

Objective 1: Combine the common Lyapunov function method with the (average) dwell time method to analyze the stability of the time-invariant switched linear system, which evolves upon a locally overlapped switching law.

Objective 2: Validate the stability analysis method via an example and state an application to flight controller design.

III. Basic Results

Combining the common Lyapunov function method and the dwell time method, a new stability analysis tool is deduced in this section and it is validated via a simple example.

Consider the time-invariant switched linear system

$$\dot{x}(t) = A_{\sigma(t)} x(t) \tag{7}$$

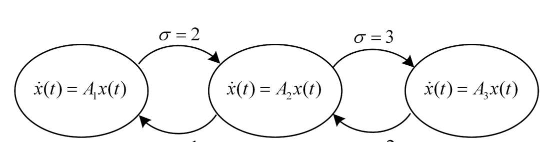


Fig. 1 Switched system with locally overlapped switching law.

where the system matrices A_i are all Hurwitz for every $i \in \Omega$, the switching law $\sigma(t) \to \Omega$ is localizable, and thereby the system (7) can be rewritten in a localizable formulation:

$$\begin{cases} \dot{x}(t) = A_{\sigma_1(t)}x(t), & \sigma_1(t) \to \Omega_1 \\ \dot{x}(t) = A_{\sigma_2(t)}x(t), & \sigma_2(t) \to \Omega_2 \\ \qquad \vdots \\ \dot{x}(t) = A_{\sigma_k(t)}x(t), & \sigma_k(t) \to \Omega_k \end{cases} \tag{8}$$

Let $S[\tau_d, \mathfrak{J}]$ denote the class of all the locally overlapped switching laws with the property that the duration of being in the subset $\mathfrak{J}$-based common subsystem is no smaller than $\tau_d > 0$. Let $\rho_{\max}(\bullet)$, $\rho_{\min}(\bullet)$ denote the largest and the smallest singular value, respectively. $\lambda_{\min}(\bullet)$ denotes the smallest eigenvalue.

A. Stability Analysis

Theorem 1: Given the family of systems (8), suppose that there exist appropriately dimensioned positive symmetric matrices P_j, Q_{ij} satisfying the condition that $A_i^T P_j + P_j A_i = -Q_{ij}$ for every $j \in \sim^k$, $i \in \Omega_j$, and the switching law $\sigma(t) \in S[\tau_d, \mathfrak{J}]$ and τ_d, called dwell time, satisfy the inequality

$$\tau_d > \frac{\ell n\, \mu}{\inf\limits_{j\in\mathfrak{J}, i\in\Omega^c_{\mathfrak{J}}}[\lambda_{\min}(P_j^{-1}Q_{ij})]}, \qquad \mu = \sup_{j,l\in\mathfrak{J}}\left[\frac{\rho_{\max}(P_j)}{\rho_{\min}(P_l)}\right] \tag{9}$$

Then the system (8) is globally asymptotically stable.

Proof: Take the following Lyapunov functions for every locally overlapped switched system in (8):

$$V_j(t) = x^T(t)P_j x(t) \tag{10}$$

Because $A_i^T P_j + P_j A_i = -Q_{ij}$ holds for every $j \in \sim^k$, $i \in \Omega_j$, $V_j(t)$ is a common Lyapunov function, that is, under an arbitrary subswitching law $\sigma_j(t)$, the value of $V_j(t)$ corresponding to the jth LOSS is monotonically decreasing.

In what follows, we concentrate our focus on the situation in which switching occurs between the LOSSs.

Without loss of generality, suppose that the LOSSs indexed by p and q are consecutively activated when time elapses. Let t_p denote the time when the subset $\{p, q\}$-based common subsystem is activated while t_q denotes the inactivated time. Apparently, if $V_p(t_p^-) < V_q(t_q^+)$ holds, it implies that the value of the Lyapunov function is increasing during the time of the common subsystem being active, that is, switching between the LOSSs makes the value of the Lyapunov function increase. Thus, we should ensure that $V_q(t_q^+) < V_p(t_p^-)$ holds and this inequality can be viewed as a sufficient condition for the globally asymptotical stability. The aforementioned illustration is also depicted in Fig. 2, wherein the value of Lyapunov function at point B is smaller than that of point A.

Assuming that t^* is the duration time in the subset $\{p, q\}$-based common subsystem, then we have

$$V_q(t_q^+) < V_q(t_p^+)e^{-\eta_q t^*} \tag{11}$$

where $\eta_q \triangleq \inf_{t\geq t_q}(-\frac{\dot{V}_q(t)}{V_q(t)})$ is the decay rate of the Lyapunov function $V_q(t)$. It is easy to deduce that $\eta_q = \inf_{i\in\Omega^c_{\{p,q\}}}[\lambda_{\min}(P_q^{-1}Q_{iq})]$ [14], such that

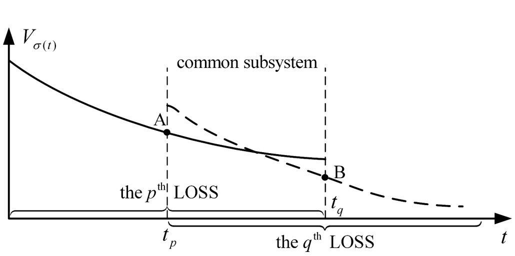


Fig. 2 Switching between two LOSSs.

$$V_q(t_q^+) < V_q(t_p^+)e^{-\inf_{i\in\Omega^c_{\{p,q\}}}[\lambda_{\min}(P_q^{-1}Q_{iq})]t^*} \tag{12}$$

Considering $\mu = \sup_{j,l\in\{p,q\}}[\rho_{\max}(P_j)/\rho_{\min}(P_l)]$, thus,

$$V_q(t_p^+) \leq \mu V_p(t_p^-) \tag{13}$$

By Eqs. (12) and (13), we conclude that

$$V_q(t_q^+) < \mu e^{-\inf_{i\in\Omega^c_{\{p,q\}}}[\lambda_{\min}(P_q^{-1}Q_{iq})]t^*} V_p(t_p^-) \tag{14}$$

Because of $\sigma(t) \in S[\tau_d, \mathfrak{J}]$, we have $t^* \geq \tau_d > \frac{\ell n\, \mu}{\inf_{j\in\mathfrak{J}, i\in\Omega^c_{\mathfrak{J}}}[\lambda_{\min}(P_j^{-1}Q_{ij})]}$; it yields that $\mu e^{-\inf_{i\in\Omega^c_{\{p,q\}}}[\lambda_{\min}(P_q^{-1}Q_{iq})]t^*} < 1$, namely,

$$V_q(t_q^+) < V_p(t_p^-) \tag{15}$$

This means that the value of the Lyapunov function of the switched system is monotonically decreasing when switching occurs between the LOSSs. Because each LOSS is also monotonically decreasing under any subswitching law, we finally conclude that the switched system (8) is globally asymptotically stable. □

B. Validation

Definition 5 [15]: The matrix pencil $\Upsilon[A_i, M]$ is

$$\Upsilon[A_i, M] \triangleq \sum_{i=1}^{M} \alpha_i A_i, \qquad \alpha_i \geq 0, \qquad \sum_{i=1}^{M} \alpha_i > 0 \tag{16}$$

Lemma 1 [15]: A sufficient condition for the existence of a switching law, such that system (7) is unstable, is that there exist nonnegative constants $\alpha_i \geq 0$, $i \in \sim^M$, such that $\Upsilon[A_i, M]$ has an eigenvalue with a positive real part.

Consider the second-order switched system with the three subsystems given by Eq. (5) evolving upon a locally overlapped switching law $\sigma(t)$, which is depicted in Fig. 1. Let

$$A_1 = \begin{bmatrix} -1 & 0 \\ 5 & -1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -2 & 1 \\ 3 & -2 \end{bmatrix}$$
$$A_3 = \begin{bmatrix} -2 & 2 \\ 1 & -2 \end{bmatrix}, \qquad x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{17}$$

Setting constant $\alpha_i = 1$, $i = 1, 2, 3$, we can easily verify the matrix $\alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3$ is not Hurwitz. Using Lemma 1, we directly know that there is a switching law making the system (5) unstable, which indicates that the common Lyapunov function method is not valid in this case. We set a locally overlapped switching law $\sigma(t)$ as Eq. (18), for example, where $\mathrm{mod}(a, b)$ denotes the remainder of a divided by b and the trajectories of the states are depicted in Fig. 3.

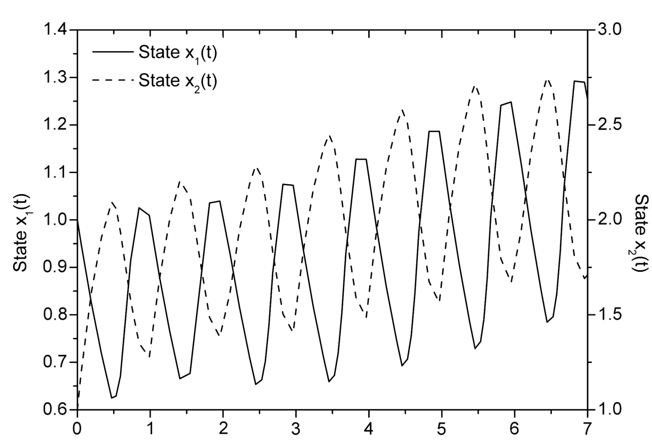


Fig. 3 Switched system unstable under locally overlapped switching law.

$$\sigma(t) = \begin{cases} 1, & \bmod(t,1) \in [0, 0.5) \\ 2, & \bmod(t,1) \in [0.5, 0.6) \\ 3, & \bmod(t,1) \in [0.6, 0.9) \\ 2, & \bmod(t,1) \in [0.9, 1) \end{cases} \tag{18}$$

In the following content, we use Theorem 1 to analyze the stability of system (5) with A_i given by Eq. (17).

As mentioned in Sec. II.B, the switched system (5) can be partitioned into two LOSSs described by Eq. (6) with the two subswitching laws $\sigma_j(t)$, $j = 1, 2$, and $\dot{x}(t) = A_2 x(t)$ is the subset $\{1, 2\}$-based common subsystem.

According to Theorem 1, let

$$P_1 = \begin{bmatrix} 3870.7 & 0 \\ 0 & 619.3 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 3758.1 & 27.4 \\ 27.4 & 2591.5 \end{bmatrix} \tag{19}$$

We achieve $\tau_d > 6.11$ by Eq. (9), which means that switched system (5) is globally asymptotically stable if the duration of being in the subset $\{1, 2\}$-based common subsystem is larger than 6.11.

Set locally overlapped switching law $\sigma(t)$ as follows, for example,

$$\sigma(t) = \begin{cases} 1, & \bmod(t,13) \in [0, 0.2) \\ 2, & \bmod(t,13) \in [0.2, 6.4) \\ 3, & \bmod(t,13) \in [6.4, 6.7) \\ 2, & \bmod(t,13) \in [6.7, 13) \end{cases} \tag{20}$$

The trajectories of the states are depicted in Fig. 4, which indicates that our stability analysis tool is quite efficient. Besides, note that the stability analysis method permits arbitrary fast switching between subsystems of each LOSS.

Remark 3: The dwell time method can also be used to analyze the stability of such a switched system, but it deems to be so conservative that the interval of every consecutive switching must be no smaller than the dwell time τ_d.

IV. Extended Results

In the sequel, we will extend the stability analysis method obtained in Sec. III to combination of the common Lyapunov function method and the average dwell time method. Extended result only restricts limited numbers of switching between a finite time interval during which some common subsystems are active. This extended result allows the possibility of switching fast when necessary and compensates for it by switching sufficiently slow later on for stability.

Let $\Delta T > 0$ denote the work time during which several common subsystems belonging to different LOSSs are active, and we let $N_\sigma(\Delta T)$ denote the number of switching of $\sigma(t)$ over ΔT. For given $N_0 > 0$, $\tau_a > 0$, $S[\tau_a, N_0]$ denotes the set of all switching laws satisfying

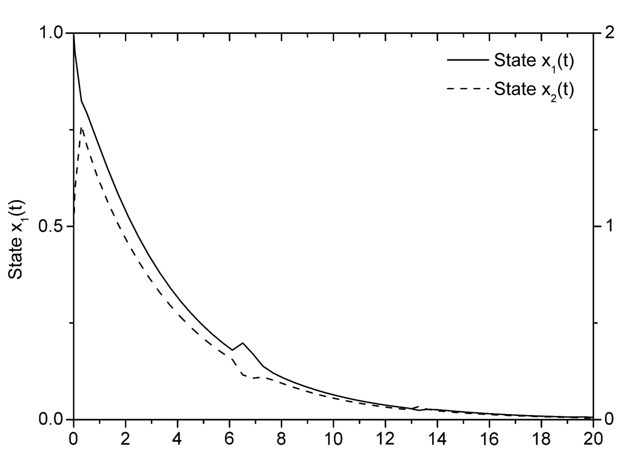


Fig. 4 Switched system stable under locally overlapped switching law.

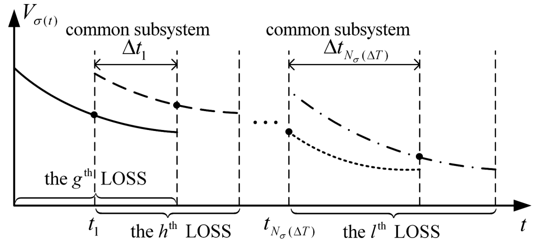


Fig. 5 Switching in LOSSs with the constraint of average dwell time.

$$N_\sigma(\Delta T) \le N_0 + \frac{\Delta T}{\tau_a}, \qquad \forall \Delta T \ge 0 \tag{21}$$

where τ_a is called average dwell time [9].

A. Stability Analysis

Theorem 2: Given the family of systems (8), suppose that there exist appropriately dimensioned positive symmetric matrixes P_j, Q_{ij} satisfying the condition $A_i^T P_j + P_j A_i = -Q_{ij}$ for any $j \in \sim^k$, $i \in \Omega_j$. Suppose also that $\sigma(t) \in S[\tau_a, N_0]$ and average dwell time τ_a satisfies the inequality

$$\tau_a > \frac{\ln \mu}{\inf_{j\in\mathfrak{J}, i\in\Omega^c_{\mathfrak{J}}} [\lambda_{\min}(P_j^{-1} Q_{ij})]}, \qquad \mu = \sup_{j,l\in\mathfrak{J}} \left[\frac{\rho_{\max}(P_j)}{\rho_{\min}(P_l)}\right] \tag{22}$$

Then the system (8) is globally asymptotically stable.

Proof: Let $t_1, t_2, \ldots, t_{N_\sigma(\Delta T)}$ denote the switching times when certain common subsystems belonging to different LOSSs are consecutively activated, and let $\Delta t_1, \Delta t_2, \ldots, \Delta t_{N_\sigma(\Delta T)}$ denote the duration time of each common subsystem being active; thereby, we know that $\Delta T = \Delta t_1 + \Delta t_2 + \cdots \Delta t_{N_\sigma(\Delta T)}$, which is also depicted in Fig. 5.

For every locally overlapped switched system in Eq. (8), we take the Lyapunov functions in form (10) and thereby it is easy to obtain that each LOSS is monotonically decreasing under an arbitrary subswitching law $\sigma_j(t)$. Similar to the deduction of Theorem 1, we concentrate our focus on the case in which switching occurs between the LOSSs.

Similar to the illustration in the Proof of Theory 1, we can conclude that the following inequality

$$V_l((t_{N_\sigma(\Delta T)} + \Delta t_{N_\sigma(\Delta T)})^+) < V_g(t_1^-), \quad \forall\ g, l \in \sim^k \tag{23}$$

is a sufficient condition for the stability when switching occurs between the LOSSs, which is also depicted in Fig. 5 wherein the value of Lyapunov function at the time $t_{N_\sigma(\Delta T)} + \Delta t_{N_\sigma(\Delta T)}$ is smaller than that of t_1.

For two consecutively activated LOSSs indexed by g and h, it holds that

$$V_h((t_1 + \Delta t_1)^+) < V_h(t_1^+) e^{-\eta_h \Delta t_1} \le \mu V_g(t_1^-) e^{-\eta_h \Delta t_1} \tag{24}$$

Iterating this technique from 1 to $N_\sigma(\Delta T)$ and by Eq. (21), we can obtain

$$\begin{aligned} V_l((t_{N_\sigma(\Delta T)} + \Delta t_{N_\sigma(\Delta T)})^+) &< \mu^{N_\sigma(\Delta T)} V_g(t_1^-) e^{-\eta_h \Delta t_1 - \cdots - \eta_l \Delta t_{N_\sigma(\Delta T)}} \\ &< \mu^{N_\sigma(\Delta T)} V_g(t_1^-) e^{-\inf_{j\in\mathfrak{J}, i\in\Omega^c_{\mathfrak{J}}} [\lambda_{\min}(P_j^{-1} Q_{ij})](\Delta t_1 + \cdots + \Delta t_{N_\sigma(\Delta T)})} \\ &< \mu^{N_\sigma(\Delta T)} V_g(t_1^-) e^{-\inf_{j\in\mathfrak{J}, i\in\Omega^c_{\mathfrak{J}}} [\lambda_{\min}(P_j^{-1} Q_{ij})]\Delta T} \\ &< V_g(t_1^-) e^{(N_0 + \frac{\Delta T}{\tau_a}) \ln \mu} e^{-\inf_{j\in\mathfrak{J}, i\in\Omega^c_{\mathfrak{J}}} [\lambda_{\min}(P_j^{-1} Q_{ij})]\Delta T} \\ &< V_g(t_1^-) e^{N_0 \ln \mu} e^{(\frac{\ln \mu}{\tau_a} - \inf_{j\in\mathfrak{J}, i\in\Omega^c_{\mathfrak{J}}} [\lambda_{\min}(P_j^{-1} Q_{ij})])\Delta T} \end{aligned} \tag{25}$$

By the inequation (22) on average dwell time, we derive the inequation $\frac{\ln \mu}{\tau_a} - \inf_{j\in\mathfrak{J}, i\in\Omega^c_{\mathfrak{J}}} [\lambda_{\min}(P_j^{-1} Q_{ij})] < 0$, which implies that

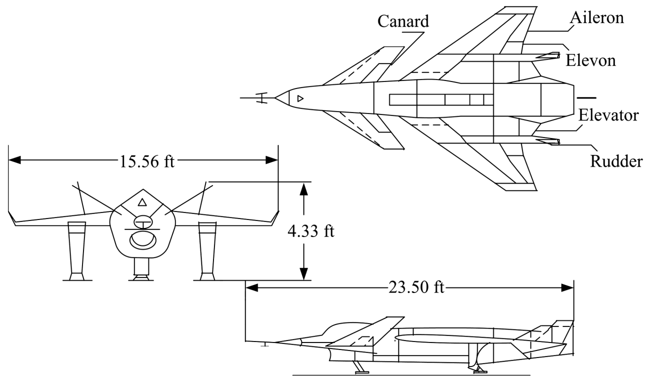


Fig. 6 Three-view drawing of the HiMAT vehicle.

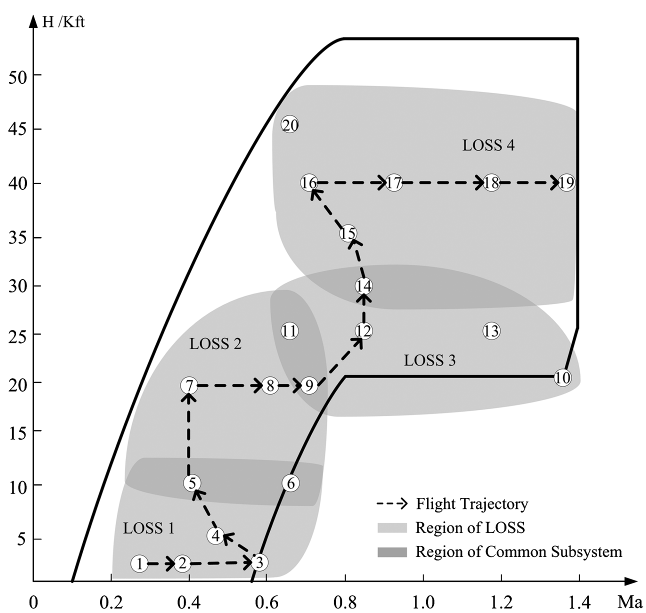


Fig. 7 Flight envelope of the HiMAT vehicle.

there must exist $\Delta T^* > 0$ such that the inequation (23) holds when $\Delta T \to +\infty$.

This means that the value of the Lyapunov function of the switched system is monotonically decreasing when the switching occurs between the LOSSs. Because the LOSS is monotonically decreasing under any subswitching law, we conclude that the switched system (8) is globally asymptotically stable. Remark that if ΔT is finite, namely, no common subsystem is activated for any $t > t_{N_\sigma(\Delta T)} + \Delta t_{N_\sigma(\Delta T)}$, then the system (8) evolving on a certain subswitching law and the corresponding subsystems share a common Lyapunov function such that the system (8) is also globally asymptotically stable. □

B. Application to Full-Envelope Flight Controller Design

In practice, most of full-envelope flight control systems are designed by the gain-scheduled method. The gain-scheduled method needs the parameters of the plant to vary slowly, but for most of modern aircraft, especially for hypersonic vehicles and high-performance fighters, the parameters vary quickly such that the gain-scheduled method is no longer theoretically feasible. In view of the statements herein, it is quite important to orchestrate a control scheme that satisfies the analysis and synthesis demands with the fast-varying parameters. Switched systems meet the aforementioned demands, as parameter varying can be naturally described by the switched system evolving upon arbitrary fast switching law. In addition, it is interesting that the asymptotical stability for the switched linear system under an arbitrary switching law is equivalent to the robust asymptotical stability for a polytopic uncertain linear time-variant system for which vertex dynamics are the subsystems of the switched linear system [16]. Such a condition allows us to analyze polytopic uncertain linear time-variant system with an infinite number of modes, which is complex but highly approximates the practical dynamics of the plant (e.g., aircraft dynamics within the full flight envelope), from the view of the time-invariant switched linear system.

In what follows, an application to a full-envelope longitudinal flight controller of the highly maneuverable technology (HiMAT) vehicle is presented to demonstrate the main results presented in this section. The highly maneuverable technology research is sponsored by NASA and the U.S. Air Force. The HiMAT vehicle, an open-loop unstable aircraft, is studied to incorporate technological advances in many fields, such as a close-coupled canard configuration and advanced transonic aerodynamics. A three-view drawing of the HiMAT vehicle is given in Fig. 6 [17].

In our work, the analysis is based on the 20 operating points within the full flight envelope presented in [17], as depicted in Fig. 7. The trim condition for every operating point is illustrated in Table 1. In view of the stability and maneuverability of the vehicle primarily depending on the short period motion, we constructed a switched linear system using longitudinal short period models of the 20 operating points. Suppose that each model can describe the dynamics in the vicinity of the operating point and the 20 models cover the dynamic behaviors of the HiMAT vehicle within the full flight envelope. The switched system is modeled as follows:

$$\dot{x}(t) = A_\sigma x(t) + B_\sigma u(t), \qquad x(0) = (0,0)^T \tag{26}$$

where $x = (\alpha, q)^T$, and α and q denote the angle of attack and the pitch rate, respectively. $u = (\xi_e, \xi_v, \xi_c)^T$, where ξ_e, ξ_v, and ξ_c denote the elevator, the elevon, and the canard deflection, respectively. $\sigma(t) \to \Omega = \{1, 2, \ldots, 20\}$ evolves upon the Mach number and altitude. System and input matrices in system (26) can be obtained in [17] and omitted here.

Because the Mach number and altitude could not vary discretely, the switching law is localizable. With Fig. 7 and the data in Table 1, we can easily partition the switched system, which is composed of the 20 subsystems on 20 operating points, into four LOSSs. LOSS 1 is composed of the subsystems on operating points 1, 2, 3, 4, 5, and 6; LOSS 2 contains the subsystems on 5, 6, 7, 8, 9, and 11; LOSS 3 contains the subsystems on 9, 10, 11, 12, 13, and 14; and LOSS 4 is composed of 14, 15, 16, 17, 18, 19, and 20. Thereby, the subsystems on 5 and 6 are common subsystems shared by LOSS 1 and LOSS 2, the subsystems on 9 and 11 are common subsystems shared by LOSS 2 and LOSS 3, and the subsystem on 14 is the common subsystem shared by LOSS 3 and LOSS 4. This statement is also depicted in Fig. 7.

Table 1 Twenty operating points of HiMAT vehicle

Operating point	Mach number	Altitude, ft	Angle of attack, deg
1	0.29	2500	3.18
2	0.4	2500	1.49
3	0.6	2500	0.69
4	0.5	5000	1.02
5	0.4	10,000	2.17
6	0.7	10,000	0.73
7	0.4	20,000	3.60
8	0.6	20,000	1.48
9	0.7	20,000	1.08
10	1.4	20,000	2.06
11	0.7	25,000	1.38
12	0.9	25,000	1.19
13	1.2	25,000	2.15
14	0.9	30,000	1.36
15	0.8	35,000	1.77
16	0.7	40,000	2.98
17	0.9	40,000	1.96
18	1.2	40,000	2.23
19	1.4	40,000	2.03
20	0.68	45,000	4.11

We set the state feedback control law $u(t) = K_\sigma(r(t) - x(t))$ such that the eigenvalues of the closed-loop system

$$\dot{x}(t) = (A_\sigma - B_\sigma K_\sigma)x(t) + B_\sigma K_\sigma r(t), \qquad x(0) = (0,0)^T \quad (27)$$

equal $-4.5 \pm 4.5i$ for $i \in \{1, 2, \ldots, 9\}$ and $-3.92 \pm 4i$ for $i \in \{10, 11, \ldots, 20\}$. $r = (\alpha_c, q_c)^T$ is the command input, including angle of attack command and pitch rate command. The controller gains are listed in as follows:

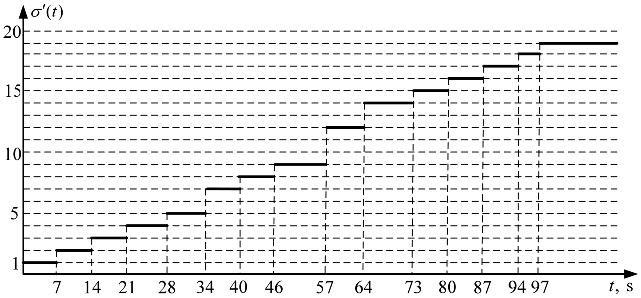


Fig. 8 Switching law within the full flight envelope.

$$K_1 = \begin{bmatrix} -3.4302 & -0.46719 \\ -2.3647 & -0.32207 \\ 0.30393 & 0.041394 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -2.1457 & -0.20713 \\ -1.4623 & -0.14116 \\ 0.23128 & 0.022326 \end{bmatrix}, \quad K_3 = \begin{bmatrix} -1.3015 & -0.05367 \\ -0.79935 & -0.03296 \\ -0.14162 & -0.00584 \end{bmatrix}$$

$$K_4 = \begin{bmatrix} -1.6262 & -0.1249 \\ -1.0918 & -0.0839 \\ 0.00647 & 0.000498 \end{bmatrix}, \quad K_5 = \begin{bmatrix} -2.6126 & -0.32122 \\ -1.794 & -0.22058 \\ 0.2617 & 0.032177 \end{bmatrix}, \quad K_6 = \begin{bmatrix} -0.96854 & -0.057254 \\ -0.64002 & -0.037834 \\ -0.03540 & -0.002093 \end{bmatrix}$$

$$K_7 = \begin{bmatrix} -3.6816 & -0.55059 \\ -2.5446 & -0.38055 \\ 0.31356 & 0.046892 \end{bmatrix}, \quad K_8 = \begin{bmatrix} -1.7093 & -0.19408 \\ -1.1066 & -0.12565 \\ -0.1632 & -0.01853 \end{bmatrix}, \quad K_9 = \begin{bmatrix} -1.1146 & -0.1170 \\ -0.7867 & -0.083 \\ 0 & 0 \end{bmatrix}$$

$$K_{10} = \begin{bmatrix} 2.1523 & -0.04683 \\ 0.73951 & -0.01609 \\ -0.0526 & 0.001144 \end{bmatrix}, \quad K_{11} = \begin{bmatrix} -1.0763 & -0.13074 \\ -0.73979 & -0.08986 \\ -0.03037 & -0.00369 \end{bmatrix}, \quad K_{12} = \begin{bmatrix} -0.4622 & -0.04562 \\ -0.3234 & -0.0319 \\ 0.01101 & 0.00109 \end{bmatrix}$$

$$K_{13} = \begin{bmatrix} 1.0307 & -0.04870 \\ 0.4304 & -0.02034 \\ -0.0252 & 0.00119 \end{bmatrix}, \quad K_{14} = \begin{bmatrix} -0.5035 & -0.06888 \\ -0.3575 & -0.04890 \\ 0.01380 & 0.0019 \end{bmatrix}, \quad K_{15} = \begin{bmatrix} -1.0467 & -0.1627 \\ -0.7513 & -0.11681 \\ 0.0547 & 0.008504 \end{bmatrix}$$

$$K_{16} = \begin{bmatrix} -1.9656 & -0.3290 \\ -1.3705 & -0.2294 \\ -0.0402 & -0.0067 \end{bmatrix}, \quad K_{17} = \begin{bmatrix} -0.7522 & -0.13986 \\ -0.5465 & -0.10161 \\ 0.02364 & 0.00440 \end{bmatrix}, \quad K_{18} = \begin{bmatrix} 0.84827 & -0.15284 \\ 0.47806 & -0.08614 \\ -0.0326 & 0.00587 \end{bmatrix}$$

$$K_{19} = \begin{bmatrix} 1.1865 & -0.14384 \\ 0.56662 & -0.06869 \\ 0.00557 & -0.00068 \end{bmatrix}, \quad K_{20} = \begin{bmatrix} -2.5462 & -0.4791 \\ -1.7621 & -0.3315 \\ -0.0773 & -0.0145 \end{bmatrix} \quad (28)$$

According to Theorem 2, let

$$P_1 = \begin{bmatrix} 0.50438 & -0.024478 \\ -0.024478 & 0.030154 \end{bmatrix}$$
$$P_2 = \begin{bmatrix} 0.63491 & -0.024711 \\ -0.024711 & 0.032889 \end{bmatrix}$$
$$P_3 = \begin{bmatrix} 0.77628 & -0.043016 \\ -0.043016 & 0.054003 \end{bmatrix}$$
$$P_4 = \begin{bmatrix} 0.91996 & -0.025222 \\ -0.025222 & 0.045408 \end{bmatrix} \quad (29)$$

we have $\tau_a > 6.9315$ by the inequality (22), which implies that the switched system

$$\dot{x}(t) = (A_\sigma - B_\sigma K_\sigma)x(t) \quad (30)$$

is globally (uniformly) asymptotically stable under any switching law $\sigma(t) \in S[\tau_a, N_0]$. It should be noted that the uniform property is,

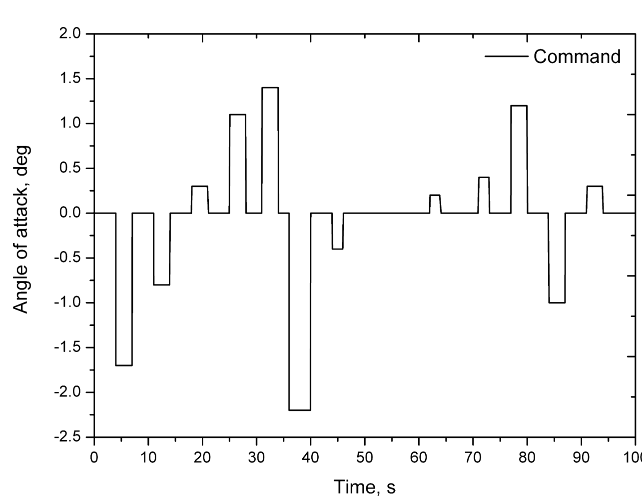
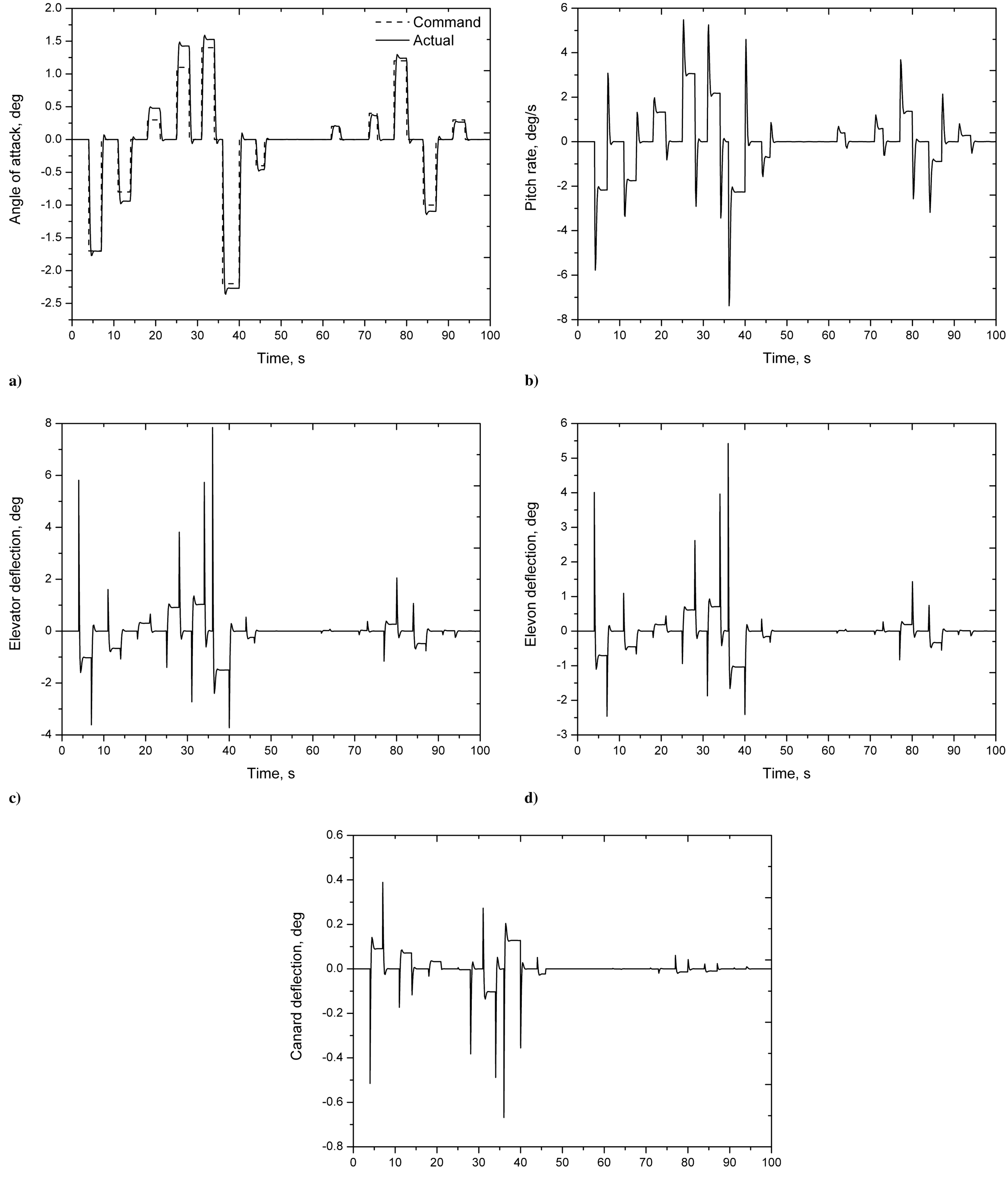


Fig. 9 Angle of attack command.



e)

Fig. 10 Trajectories of system states and control inputs: a) trajectory of angel of attack, b) trajectory of pitch rate, c) elevator deflection, d) elevon deflection, and e) canard deflection.

in this sense, over any $\sigma(t) \in S[\tau_a, N_0]$ but not over any initial time for time-varying systems [2]. Further, because the globally uniformly asymptotical stability means the globally uniformly exponential stability for switched linear systems [2], by Lemma 1 in [9], we conclude that the closed-loop switched system (27) is input-to-state uniformly bounded under any switching law $\sigma(t) \in S[\tau_a, N_0]$, that is, the system state is uniformly bounded for any piecewise continuous command input $r = (\alpha_c, q_c)^T$.

In the sequel, we simulate the system response with the switching law $\sigma'(t)$ depicted in Fig. 8 and set $N_0 = 2$.

The switching law denotes a large-scale flight trajectory, that is, 1-2-3-4-5-7-8-9-12-14-15-16-17-18-19, which is also depicted in Fig. 7. Because inequality (21) holds for every time interval in common subsystems, for example, the total time interval in common subsystems 5, 9, and 14 equals 26 s (6 s + 11 s + 9 s) and 26 s/6.9315 s + $N_0 = 5.7510 > 3$, we have $\sigma'(t) \in S[\tau_a, N_0]$ so that the closed-loop switched system (27) is input-to-state uniformly bounded.

It should be noted that discontinuous changes of switching law $\sigma'(t)$ do not imply that Mach number or altitude varies discontinuously; in fact, Mach number and altitude vary continuously in this study. Because we suppose each subsystem can describe the dynamics in the vicinity of the operating point, the switching action arises at the time when the vehicle flies into another vicinity of an operating point, which makes the switching law vary abruptly.

The pitch rate command is set to be zero. The angle of attack command is depicted in Fig. 9, and it can picture the flight trajectory to some extent. For example, the angle of attack trims at 3.18 and 1.49 deg for operating points 1 and 2, respectively; if the HiMAT vehicle flies into the vicinity of operating point 2 from 1 at 7 s, the angle of attack should be commanded to decrease about 1.7 deg before 7 s (e.g., a 1.7 deg downward step at 4 s) and reset to zero when the vehicle flies into the vicinity of operating point 2.

The simulation results are depicted in Figs. 10a–10e.

By the figures, we can conclude that the angle-of-attack tracking performance is acceptable along the whole flight trajectory, and the pitch rate response and control surface deflection are all satisfying over the entire time history, even when the system dynamics vary quickly during the time interval 90–97 s.

Other large-scale flight trajectories within the full flight envelope, for example, 3-6-9-12-14-15-16-20, are validated by simulation as well and a similar conclusion can be achieved, which demonstrates the stability of the switched system (corresponding to the dynamic behaviors of the HiMAT vehicle) with a locally overlapped switching law (evolving upon Mach number and altitude) and shows the effectiveness of the proposed stability analysis tool.

V. Conclusions

Switched systems with a locally overlapped switching law exist in many practical engineering problems. In this study, we propose stability analysis tools for such switched systems by combining the common Lyapunov function method and the (average) dwell time method. We show that the switched linear system is globally asymptotically stable provided that the (average) dwell time on common subsystems belonging to finite locally overlapped switched systems is no smaller than a fixed positive constant. Future works will analyze the impact of system nonlinearities or disturbances and address the application of this methodology to other systems, such as separation dynamical systems and boundary control systems.

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